June 29, 1728: "Je finirai par un problème qui m'a paru fort curieux et que j'ai résolu. Le voici: Trouver deux nombres inégaux x et y tels que $x^y = y^x$. Il n'y a qu'un cas ou ces nombres soient entiers, savoir $x = 2$ et $y = 4$ (car $2^4 = 4^2$), mais on peut donner une infinité de nombres rompus qui satisfont au problème. Il y a aussi d'autres espèces de quantités dont je ne dirai rien."[1] In reply to this on January 31, 1729, Goldbach wrote as follows[2] (*l.c.* pp. 280–281): "Je ne trouve pas la moindre difficulté à faire voir que, dans l'équation $x^y = y^x$, les nombres x et y ne peuvent être entiers à moins que l'un ne soit = 2, et l'autre = 4, et que, pour les nombres rompus, on peut donner une infinité de solutions. Voici comment je m'y prends: Je fais $y = ax$, donc $x^{ax} = a^x x^x$ et enfin $x = a^{1/(a-1)}$. Or, il est visible que x ne peut être un nombre entier que dans la supposition de $a = 2$; car si a est un nombre entier plus grand que 2, on voit d'abord que x devient irrationel; d'un autre côté, a étant un nombre rompu, toutes ses puissances seront autant de nombres rompus, et par conséquent x ne peut être un nombre entier; mais pour exprimer la valeur de x par des nombres rompus, il n'y a qu'à faire

$$x = f^{g/(f-g)} : g^{g/(f-g)}$$

où f et g soient des nombres entiers."[3] ARC.

PROBLEMS—SOLUTIONS

2791 [1919, 414]. A cup of wine is suspended over a cup of equal capacity full of water; through a small hole in the bottom, the wine drips into the water, and the mixture drips out at

[1] Concerning this passage Cantor remarks (*Vorlesungen über Geschichte der Mathematik*, vol. 3_2, 1901, p. 610): "Man wird nach diesem Schlussworte wohl oder übel annehmen müssen, dass Bernoulli an complexe Auflösungen dachte." One must agree with Eneström (*Bibliotheca Mathematica*, vol. 13 (3), p. 270) that this surmise is "höchst unwahrscheinlich." The natural interpretation is that Bernoulli referred to the infinite number of irrational values of x and y which are obtained from Euler's equations given above, when u is not integral.

[2] The argument of Herbst is very similar.

[3] Other discussions of the relation $x^y = y^x$ are: T. Wittstein, *Archiv der Mathematik und Physik*, vol. 6, pp. 154–162, 1845; I. L. A. Lecointe, *Nouvelles Annales de Mathématiques*, vol. 11, pp. 187–189, 1852; M. Cantor, *Zeitschrift für mathematische und naturwissenschaftlichen Unterricht*, vol. 9, pp. 163–164, 1878; L. F. Marrecas Ferreira, *Jornal de Sciencias Mathematicas e Astronomicas*, vol. 2, pp. 165–166, 1880; M. Luxemberg, *Archiv der Mathematik und Physik*, vol. 66, pp. 332–334, 1881; D. Besso, U. Danielli, and L. Carline, *Periodico di Matematica per l'Insegnamento Secondario*, vol. 5, pp. 12–15, 115–117, 117–119, 1890; A. Flechsenhaar, and R. Schimmack, *Unterrichtsblätter für Mathematik und Naturwissenschaften*, vol. 17, pp. 70–73, 1911 and vol. 18, pp. 34–35, 1912; A. Tanturri, *Periodico di Matematica* . . ., vol. 30, pp. 186–187, 1915.

In Nouvelles Annales de Mathématiques, 1876, Moret-Blanc proved (pp. 44–46) that the only positive integral solutions of the equation $x^y = y^x + 1$, are $y = 0$, x arbitrary; $y = 1$, $x = 2$; $y = 2$, $x = 3$; also Meyl proved (pp. 545–547) that the only positive integral solutions of the equation $(x + 1)^y = x^{y+1} + 1$, are $x = 0$, y arbitrary; $x = 1$, $y = 1$; $x = 2$, $y = 2$. Landau showed (*L'Intermédiaire des Mathématiciens*, 1901, pp. 151–152) that the solutions found by Moret-Blanc for the equation $x^y = y^x + 1$ are the only positive rational solutions. This equation may be obtained from the simultaneous equations $x^y = 3$, $y^x = 2$ for which E. Heis gave the approximate solution $x = 2.23925113$, $y = 1.36280365$ (*Sammlung von Beispielen und Aufgaben aus . . . Arithmetik und Algebra*. 75. Aufl., Köln, 1888, p. 372; *Ausführliche Auflösuug der in Dr. Ed. Heis' Sammlung von Biespielen enthaltenen Aufgaben*, Dritter Theil, Bonn, 1880, pp. 386–388).

In *Messenger of Mathematics*, A. Cunningham discussed the factorization of $x^y - y^x$, x and $y > 1$, and x prime to y (April, 1916, vol. 45, pp. 185–192), and of $2^x - x$, x positive (May, 1917, vol 47, pp. 1–38).

the same rate. . When the wine cup is empty, what part of the contents of the lower cup is water? [Proposed by Charles Gilpin, Jr., Philadelphia, as Problem 287 in *The Mathematical Visitor*, January, 1881, volume 1, page 193. No solution was published in the *Visitor*.]

I. Solution by C. A. Noble, University of California.

Professor Hurwitz has called attention to the fact that this problem is, except for phraseology, a special case of the note by Professor Noble in this Monthly, *1919*, 191–194. In the formula for s on page 193, if we put $s_0 = 0$, $s_1 = 1$, and $v = v_0$, we have for s (the proportion of wine) $1 - e^{-1}$, so that also the proportion of water is e^{-1}.

II. Solution by H. L. Olson, University of Michigan.

Under the hypotheses the liquid will drip from the lower cup (and hence also from the upper cup) at a constant rate, $a\sqrt{2gh}$, where a is the area of the hole, h the height of the liquid in the lower cup, and g the acceleration of a freely falling body under gravity. Let q be the capacity of each cup, and x the amount of water in the lower cup. Then

$$\frac{dx}{dt} = -\frac{a\sqrt{2gh}}{q}x.$$

Making use of the fact that when $t = 0$, $x = q$, we have

$$x = qe^{-\frac{at\sqrt{2gh}}{q}}.$$

When the upper cup is empty, $t = \dfrac{q}{a\sqrt{2gh}}$ and $x/q = e^{-1}$,

the proportion of water in the lower cup at that time.

Note.—The problem does not say that the liquid will drip from the cups at a constant rate, but only at the same rate. If it drips from the first cup under the influence of gravity the rate will not be constant. But the rate is not essential; we can assume any rate, e.g., the constant rate $a\sqrt{2gh}$, whatever a and h may mean, and get the correct result.—Editors.

III. Solution by R. E. Gaines, Richmond College.

A generalization of this problem leads to a remarkable result.

Let there be a series of cups of equal capacity full of water and arranged one below another. Pour into the first cup an equal quantity of wine at a constant rate and let the overflow in each cup go into the cup just below it. Assuming that complete mixture takes place instantaneously, find the amount of wine in each cup at any time t, and at the end of the process, at time T.

Let q denote the capacity of each cup, so that the rate of flow is q/T.

If x_K be the amount of wine in the Kth cup and x_{K-1} that in the $(K - 1)$th cup, then the rate of flow of wine into the Kth cup is x_{K-1}/T and out of it is x_K/T. Whence,

$$\frac{dx_K}{dt} = \frac{x_{K-1}}{T} - \frac{x_K}{T}.$$

Any equation in the series can be solved if the one before it has been solved. So we begin at the first cup and have

$$\frac{dx_1}{dt} = \frac{q}{T} - \frac{x_1}{T}. \quad \text{Hence, } x_1 = q\left(1 - \frac{1}{e^{t/T}}\right);$$

$$\frac{dx_2}{dt} = \frac{q\left(1 - \frac{1}{e^{t/T}}\right)}{T} - \frac{x_2}{T}. \quad \text{Hence, } x_2 = q\left(1 - \frac{1 + \frac{1}{1!}\frac{t}{T}}{e^{t/T}}\right).$$

Similarly,

$$x_3 = q\left(1 - \frac{1 + \frac{1}{1!}\frac{t}{T} + \frac{1}{2!}\left(\frac{t}{T}\right)^2}{e^{t/T}}\right);$$

and so on.

For the *final* amounts in the successive cups, we have

$$X_1 = q\left(1 - \frac{1}{e}\right), \qquad X_2 = q\left(1 - \frac{1 + \frac{1}{1!}}{e}\right), \qquad X_3 = q\left(1 - \frac{1 + \frac{1}{1!} + \frac{1}{2!}}{e}\right), \quad \cdots.$$

In general, we have

$$X_K = q\left(1 - \frac{e_K}{e}\right)$$

where e_K is the sum of the first K terms of

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \text{etc.}$$

As $K \doteq \infty$, $X_K = 0$.

Note.—Here also the rate of flow is not essential. If we take as the independent variable the amount of wine x which has been poured into the first cup, then the differential equations are

$$\frac{dx_1}{dx} + \frac{x_1}{q} = 1, \qquad \frac{dx_k}{dx} + \frac{x_k}{q} = \frac{x_{k-1}}{q}$$

and

$$x_k = q\left[1 - \left(1 + \frac{x}{q} + \frac{1}{2!}\left(\frac{x}{q}\right)^2 + \cdots + \frac{1}{(k-1)!}\left(\frac{x}{q}\right)^{k-1}\right)e^{-x/q}\right].$$

The final result is obtained by setting $x = q$.—EDITORS.

Also solved by W. D. CAIRNS, ALEXANDER KNISELY, L. C. MATHEWSON, and ARTHUR PELLETIER.

2792 [1919, 414]. Proposed by B. J. BROWN, Kansas City.

Solve the differential equation,

$$x^2(1 - x)\frac{d^2y}{dx^2} + 2x(2 - x)\frac{dy}{dx} + 2(1 + x)y = x^2.$$

SOLUTION BY C. P. SOUSLEY, Pennsylvania State College.

This equation is exact and the first integral is,

$$x^2(1 - x)\frac{dy}{dx} + x(x + 2)y = \frac{x^3 + C}{3},$$

or

$$\frac{dy}{dx} + \frac{x + 2}{x(1 - x)}y = \frac{x^3 + C}{3x^2(1 - x)}.$$

Multiplying through by the integrating factor, $x^2/(1 - x)^3$, we have

$$\frac{x^2}{(1 - x)^3}\frac{dy}{dx} + \frac{x(x + 2)}{(1 - x)^4}y = \frac{x^3 + C}{3(1 - x)^4},$$

and on integrating, we have